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A Special Hill's Equation with Discontinuous Coefficients

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Abstract

The equation

$$y'' + \lambda^2 \rho(x) y = 0$$

$$\rho(x) = 1 \qquad |x| \leq 1$$
$$= a^2 \qquad 1 < |x| < L$$

is investigated. $\rho(x)$ is taken as a periodic function of period 2L and defined as above in the interval (-L,L).

The nature of the characteristic values λ for which the equation has solutions of period L and 2L is investigated. Stability charts are drawn showing for which values of a^2 and λ all solutions are bounded, and for which some solutions are not bounded.

Table of Contents

1. Introduction.

A particular case of Hill's equation will be treated in detail, namely

$$y'' + \lambda^2 Q(x)y = 0.$$

Here $Q(x)$ will be taken as an even periodic function of period $2L$ and defined by

$$Q(x) = 1 \qquad 0 \leq x < 1$$
$$\quad = a^2 \qquad 1 \leq x \leq L.$$

Evidently $Q(x)$ is a piecewise constant function with two step discontinuities in the period interval.

2. Preliminary Remarks.

We will first quote some standard results in the theory of Hill's equation. Proofs and references may be found in [1] and [2].

When $Q(x)$ is piecewise continuous, even and has minimal period $2L$, we can find normalized solutions with initial values

$$y_1(0) = 1 \quad , \quad y_1'(0) = 0 \ ,$$
$$y_2(0) = 0 \quad , \quad y_2'(0) = 1 \ .$$

The solutions of the characteristic equation

$$\rho^2 - 2y_1(2L)\rho + 1 = 0$$

are known as characteristic exponents. According to Floquet's theorem, we

can then find solutions $f_1(x)$, $f_2(x)$ such that

$$f_1(x + 2L) = \rho_1 f_1(x)$$

$$f_2(x + 2L) = \rho_2 f_2(x)$$

when ρ_1 and ρ_2 are distinct characteristic exponents. If $\rho_1 = \rho_2$, i.e. $\rho_1 = \rho_2 = 1$ or $\rho_1 = \rho_2 = -1$, we can find a solution $y(x)$ with the property

$$y(x + 2L) = \rho y(x).$$

If $\rho = 1$, $y(x)$ has period $2L$; if $\rho = -1$, $y(x)$ has period $4L$.

Furthermore, we have a <u>stability test</u>. Solutions are stable if and only if

$$|y_1(2L)| < 1$$

or $\qquad y_1(2L) = \pm 1$ and $y_2(2L) = y_1'(2L) = 0$.

Evidently, for $|y_1(2L)| > 1$, ρ_1 and ρ_2 are real and since $\rho_1\rho_2 = 1$ at least one of them is greater than one in absolute value. The solution corresponding to that exponent evidently has increasing peak values.

Another important statement is the following: There exists a non-trivial periodic solution which is

1.) even and of period $2L$ if and only if $y_1'(L) = 0$,

2.) odd and of period $2L$ if and only if $y_2(L) = 0$,

3.) even and of period $4L$ if and only if $y_1(L) = 0$,

4.) odd and of period $4L$ if and only if $y_2'(L) = 0$.

By means of the Liouville transformation [3] we can transform equations of the form

$$f_2(x)y''(x) + f_1(x)y'(x) + [\lambda g(x) + f_o(x)]y(x) = 0$$

into the form

$$\eta''(\xi) + [\lambda + \psi(\xi)]\eta(\xi) = 0.$$

In order to effect this transformation certain differentiability conditions must be imposed on the coefficients in the equation. In particular, to apply the Licuville transformation to an equation of the type

$$y'' + \lambda^2 Q(x)y = 0$$

it is necessary that $Q(x)$ possess a second derivative. In the particular case under discussion these conditions are not fulfilled. A certain Oscillation Theorem has been proved by Haupt [2], which states that the equation

$$\eta''(\xi) + [\lambda + \psi(\xi)]\eta(\xi) = 0,$$

where $\psi(\xi)$ is a doubly differentiable, real function of period 2L, has solutions of period 2L for an infinite sequence of characteristic values $\lambda_0 < \lambda_1 < \lambda_2 < \dots$, and solutions of period 4L for a sequence $\lambda_1' < \lambda_2' < \lambda_3' < \dots$. These λ's satisfy the inequalities

$$\lambda_0 < \lambda_1' \leqq \lambda_2' < \lambda_1 \leqq \lambda_2 < \lambda_3' \leqq \lambda_4' < \lambda_3 \leqq \lambda_4 < \dots$$

and satisfy the relations

$$\lim_{n \to \infty} \lambda_n^{-1} = 0 \text{ , and}$$

$$\lim_{n \to \infty} \lambda_n'^{-1} = 0 \text{ .}$$

Furthermore, for λ's in the intervals (λ_0, λ_1'), (λ_2', λ_1), $(\lambda_2 \lambda'_3)$, (λ_4', λ_3), ... the solutions are stable , outside of these intervals unstable and at the end points they are in general unstable , except under very special conditions.

We will investigate to what extent these theorems remain valid for the equation under discussion.

3. The Calculation of the Characteristic Values for $a^2 > 0$.

We now turn to the equation

$$y''(x) + \lambda^2 y(x) = 0 \quad , \quad 0 \leqq x < 1$$

$$y''(x) + \lambda^2 a^2 y(x) = 0 \; , \quad 1 \leqq x \leqq L$$

and investigate the characteristic values under the boundary conditions

1.) $y'(0) = y'(L) = 0$ (even solutions of period 2L)
2.) $y(0) = y(L) = 0$ (odd solutions of period 2L)
3.) $y'(0) = y(L) = 0$ (even solutions of period 4L)
4.) $y(0) = y'(L) = 0$ (odd solutions of period 4L).

Corresponding to the first case, one can show by a direct calculation that

$$\begin{aligned} y(x) &= \cos \lambda x \quad , \quad 0 \leqq x \leqq 1 \\ &= \cos \lambda \cos \lambda a(x - 1) - \frac{1}{a} \sin \lambda \sin \lambda a(x - 1), \quad 1 \leqq x \leqq L. \end{aligned}$$

These solutions obviously satisfy the condition $y'(0) = 0$, and from $y'(L) = 0$ we obtain, immediately, the equation

$$a \cos \lambda \sin \lambda a(L - 1) + \sin \lambda \cos \lambda a(L - 1) = 0.$$

An equivalent form is

$$\tan \lambda = - a \tan \lambda a(L - 1).$$

In a similar manner we can construct corresponding equations for the other three cases and can write down the following set of characteristic equations:

1.) $\tan\lambda = -\, a \tan \lambda a(L - 1)$

2.) $\tan\lambda = -\,\frac{1}{a} \tan \lambda a(L - 1)$

3.) $\tan\lambda = a \cot \lambda a(L - 1)$

4.) $\tan\lambda = \frac{1}{a} \cot \lambda a(L - 1).$

We could arrive at the characteristic values of λ by starting with the characteristic exponents defined by

$$\rho^2 - 2y_1(2L)\rho + 1 = 0,$$

and asking for the values of λ for which

$$y_1(2L) = 1 \quad \text{and} \quad y_1(2L) = -1.$$

The first of these refers to the cases labeled 1.) and 2.) above and the second to the cases 3.) and 4.). A direct calculation shows that

$$y_1(2L) = \cos 2\lambda a(L - 1) \cos 2\lambda - \frac{1}{2}(a + \frac{1}{a}) \sin 2\lambda \sin 2\lambda a(L - 1)$$

and we can see that

$$y_1(2L) - 1 = -\, 2 \cos^2\lambda \cos^2\lambda a(L - 1)\left[\tan\lambda + a \tan \lambda a(L - 1)\right]$$

$$\times \left[\tan\lambda + \frac{1}{a} \tan \lambda a(L - 1)\right]$$

showing that the roots of

$$y_1(2L) - 1 = 0$$

correspond to the roots of the characteristic equations of cases 1.) and 2.).

Similarly,

$$y_1(2L) + 1 = 2\cos^2\lambda \sin^2\lambda a(L - 1)\left[\tan\lambda - a\cot\lambda a(L - 1)\right]$$

$$\times \left[\tan\lambda - \frac{1}{a}\cot\lambda a(L - 1)\right]$$

showing the factorization corresponding to cases 3.) and 4.).

The following method will be used to obtain the characteristic values. We treat case 1.) as typical and write the equation in the form

$$\lambda a(L - 1) = - \tan^{-1}\frac{1}{a}\tan\lambda.$$

In the evaluation of the inverse tangent we must take the multivalued character of the function into account. We will graph both the right side and the left side of the equation and wherever the curves intersect, we evidently have a characteristic value.

Figure 1 was drawn for $a > 1$. For $a = 1$ the wavy lines become straight and have slope -1. If we superimpose on this figure the other three cases we obtain figure 2.

Figure 2 was drawn for $a > 1$. For $a = 1$ the curves of the type 1.) and 2.) become identical straight lines of slope -1, and similarly for the curves of type 3.) and 4.). Wherever the line $\lambda a(L - 1) = f(\lambda)$ intersects these curves, we obtain characteristic values. If $\left\{\lambda_i\right\}$ denote the characteristic λ's corresponding to solutions of period 2L and $\left\{\lambda_i'\right\}$ the ones corresponding to solutions of period 4L, we find that

$$0 = \lambda_0 < \lambda_1' \leq \lambda_2' < \lambda_1 \leq \lambda_2 < \lambda_3' \leq \lambda_4' < \lambda_3 \leq \lambda_4 < \dots .$$

Thus Haupt's Oscillation Theorem remains valid.

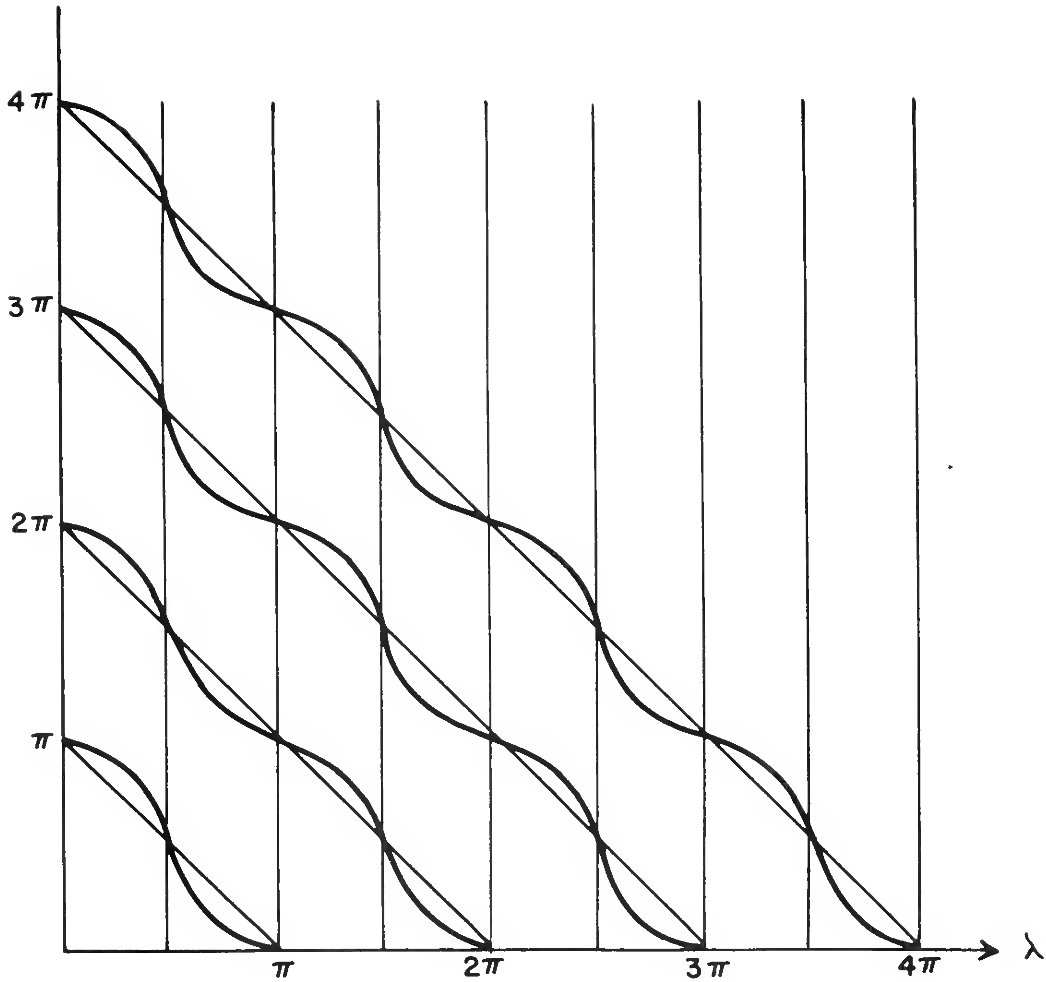

Figure 1

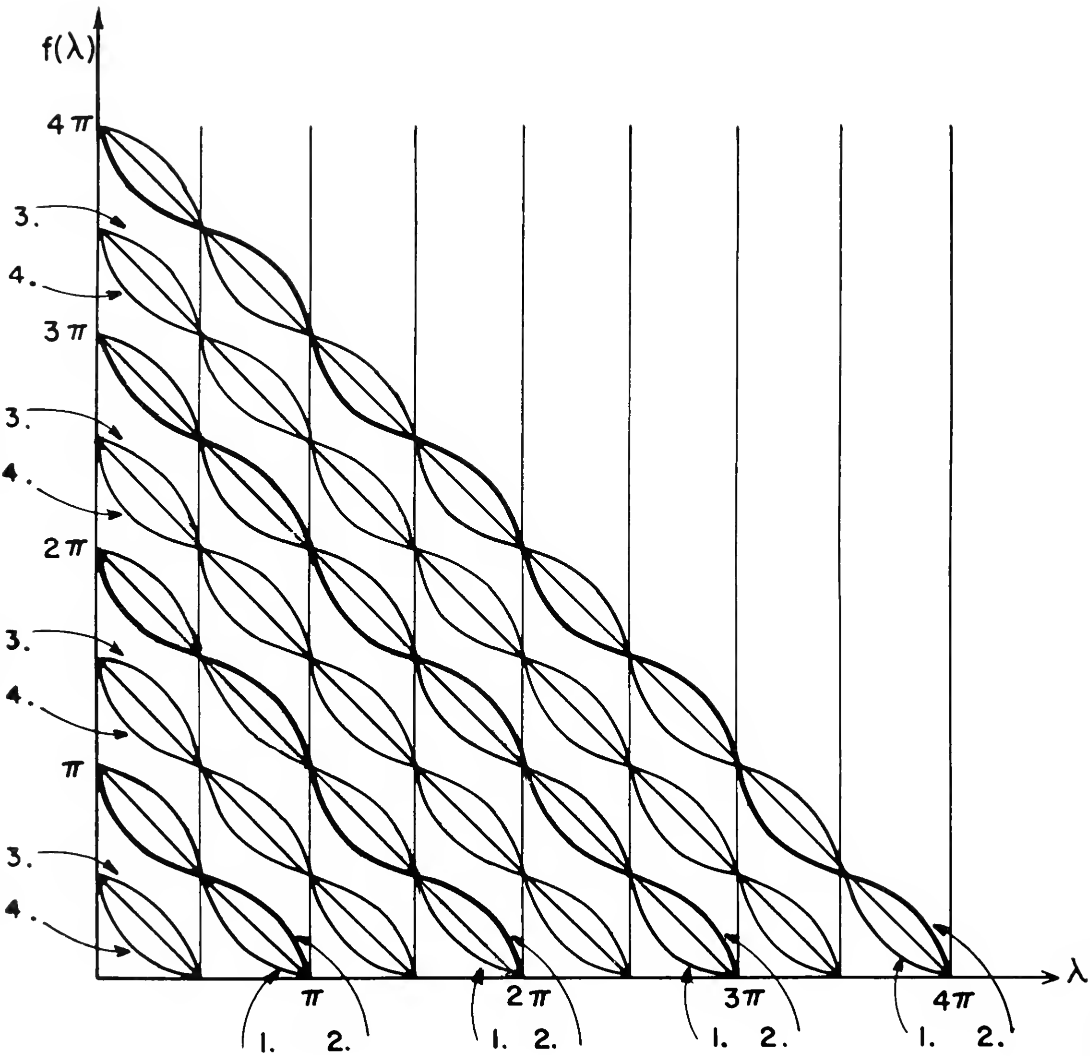

Figure 2

For $a < 1$ the curves of type 1.) and 2.) change place and similarly for types 3.) and 4.).

We can immediately determine where multiple roots occur. Evidently for $a = 1$ we have

$$\left.\begin{aligned} \lambda'_{2i+1} &= \lambda'_{2i+2} \\ \lambda_{2i+1} &= \lambda_{2i+2} \end{aligned}\right\} \qquad i = 0,1,2,\ldots .$$

In general even when $a \neq 1$, we have multiple roots for $a(L - 1)$ rational, say p/q, where p and q are relatively prime. Then we have double roots at the lattice points (kp,kq) such that for

$$\left.\begin{aligned} k(p + q) &= 2n , & \lambda_{2n-1} &= \lambda_{2n} \\ k(p + q) &= 2n + 1 , & \lambda'_{2n+1} &= \lambda'_{2n} \end{aligned}\right\} \qquad n = 1,2,\ldots .$$

It also follows that for $a(L - 1) = p/q$, we need to compute only the first p 2 roots of each kind and all others are equal to these modulo $q\pi$.

For $a(L - 1) = 1$ we can also see that since

$$\tan \lambda'_{2i+1} = a \cot \lambda'_{2i+1}$$

$$\tan \lambda'_{2i+2} = \frac{1}{a} \cot \lambda'_{2i+2}$$

it follows that

$$\tan^2\lambda'_{2i+1} \tan^2\lambda'_{2i+2} = 1$$

so that

$$\lambda'_{2i+1} + \lambda'_{2i+2} \equiv \frac{\pi}{2} \qquad \mod \pi .$$

To obtain estimates for the roots we see that for case 1.)

$$\lambda a(L - 1) = - \tan^{-1} \frac{1}{a} \tan \lambda \approx n\pi - \lambda.$$

From this we see that

$$\lambda_{2n-1} = \left[\frac{2n}{1 + a(L - 1)} \right] \frac{\pi}{2} + \theta \frac{\pi}{2}, \qquad 0 \leqq \theta < 1.$$

(Here $[x]$ represents the greatest integer less than or equal to x.) Similarly,

$$\lambda_{2n} = \left[\frac{2n}{1 + a(L - 1)} \right] \frac{\pi}{2} + \theta \frac{\pi}{2},$$

$$\lambda'_{2n-1} = \left[\frac{2n - 1}{1 + a(L - 1)} \right] \frac{\pi}{2} + \theta \frac{\pi}{2},$$

$$\lambda'_{2n} = \left[\frac{2n - 1}{1 + a(L - 1)} \right] \frac{\pi}{2} + \theta \frac{\pi}{2}.$$

To obtain better numerical estimates in particular cases we can use Newton's method.

4. The Stability Chart for $a^2 > 0$.

We can make a rough stability chart from Fig. 1. That is, we will make a graph showing the relationship between the characteristic λ's and a. These curves subdivide the first quadrant into regions of stability and instability. That is, for any λ and a from a stable region, all solutions of the differential are stable. Outside the stability regions there is only one bounded solution.

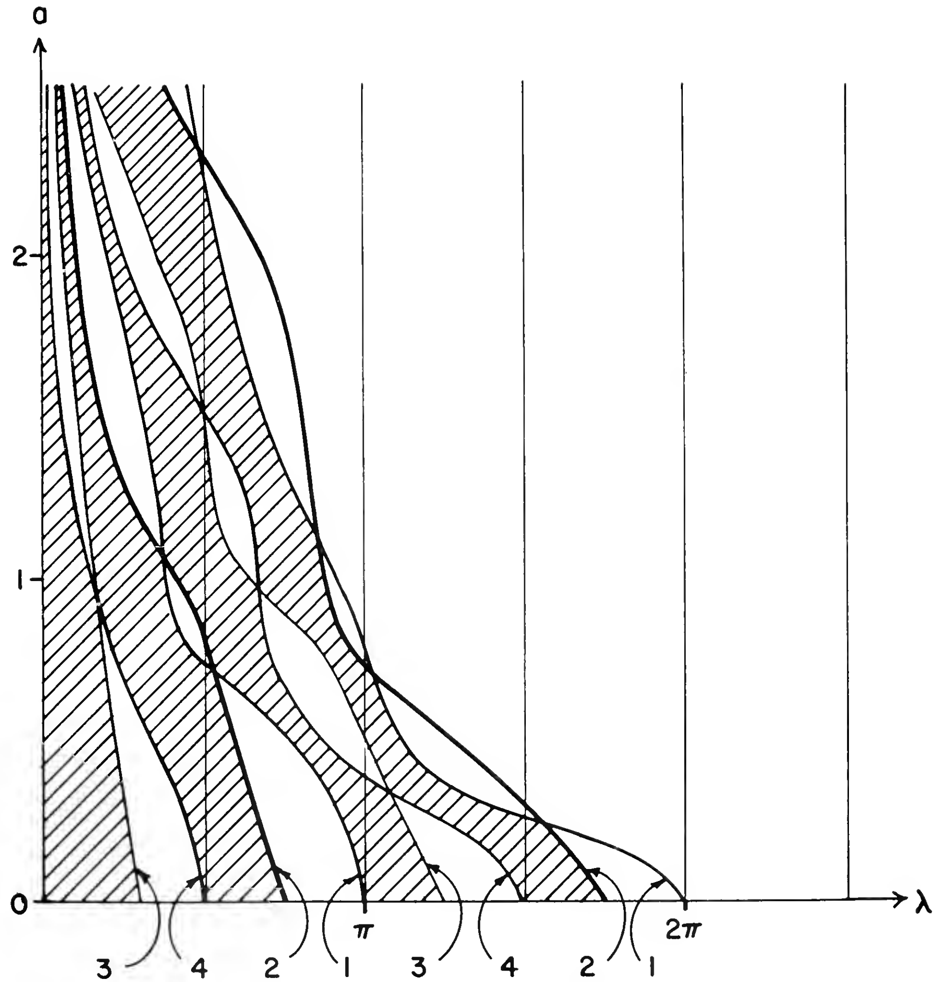

Figure 3

The shaded regions represent regions of stability. The values of λ where the curves of type 2.) and 3.) intersect the abscissa are roots of the equations

$$\tan \lambda = -\lambda(L-1)$$

$$\tan \lambda = \frac{1}{\lambda(L-1)}$$

respectively.

5. The Calculation of Characteristic Values for $a^2 < 0$.

To treat the case where $a^2 < 0$, we return to the previous equations and replace a by ia. Then we have

$$y'' + \lambda^2 y = 0 , \qquad 0 \leqq x < 1$$

$$y'' - \lambda^2 a^2 y = 0 , \qquad 1 < x \leqq L ,$$

subject to the boundary conditions

1.) $y'(0) = y'(L) = 0$

2.) $y(0) = y(L) = 0$

3.) $y'(0) = y(L) = 0$

4.) $y(0) = y'(L) = 0.$

The corresponding characteristic equations are

1.) $\tan \lambda = a \tanh \lambda a(L-1)$

2.) $\tan \lambda = -\frac{1}{a} \tanh \lambda a(L-1)$

3.) $\tan \lambda = a \coth \lambda a(L-1)$

4.) $\tan \lambda = -\frac{1}{a} \coth \lambda a(L-1).$

To evaluate the roots of these equations, we write, as before, in case 1.)

$$\lambda a(L - 1) = \tanh^{-1} \frac{1}{a} \tan \lambda$$

and graph the right side of the equation. We then obtain Figure 4, where all four cases are included.

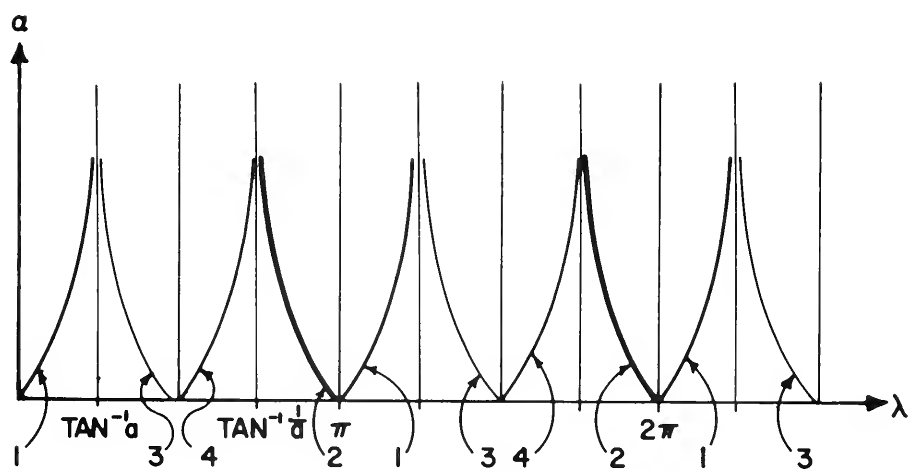

Figure 4

For $a^2(L - 1) < 1$ we again have a sequence of characteristic roots in the order given by the Oscillation Theorem

$$0 = \lambda_o < \lambda_1' < \lambda_2' < \lambda_1 < \lambda_2 < \lambda_3' < \lambda_4' < \lambda_3 < \lambda_4 < \dots .$$

In this case we obtain no double roots. For $a^2(L - 1) > 1$ a direct analysis shows that we have two roots corresponding to case 1.) in the interval $[0, \tan^{-1} a]$. Then the sequence becomes

$$0 = \lambda_o^{(1)} < \lambda_o^{(2)} < \lambda_1' < \lambda_2' < \lambda_1 < \lambda_2 < \lambda_3' < \lambda_4' < \lambda_3 < \lambda_4 < \dots .$$

For $a^2(L - 1) = 1$ $\lambda_o^{(1)} = \lambda_o^{(2)} = 0$.

For large n we see readily that

$$\lambda_{2n} \approx n\pi + \tan^{-1} a$$

$$\lambda_{2n+1}' \approx n\pi + \tan^{-1} a$$

$$\lambda_{2n+2}' \approx (n + 1)\pi - \tan^{-1} \frac{1}{a}$$

$$\lambda_{2n+1} \approx (n + 1)\pi - \tan^{-1} \frac{1}{a} .$$

Figure 5 is the stability chart for this problem, where the regions of stability are shaded.

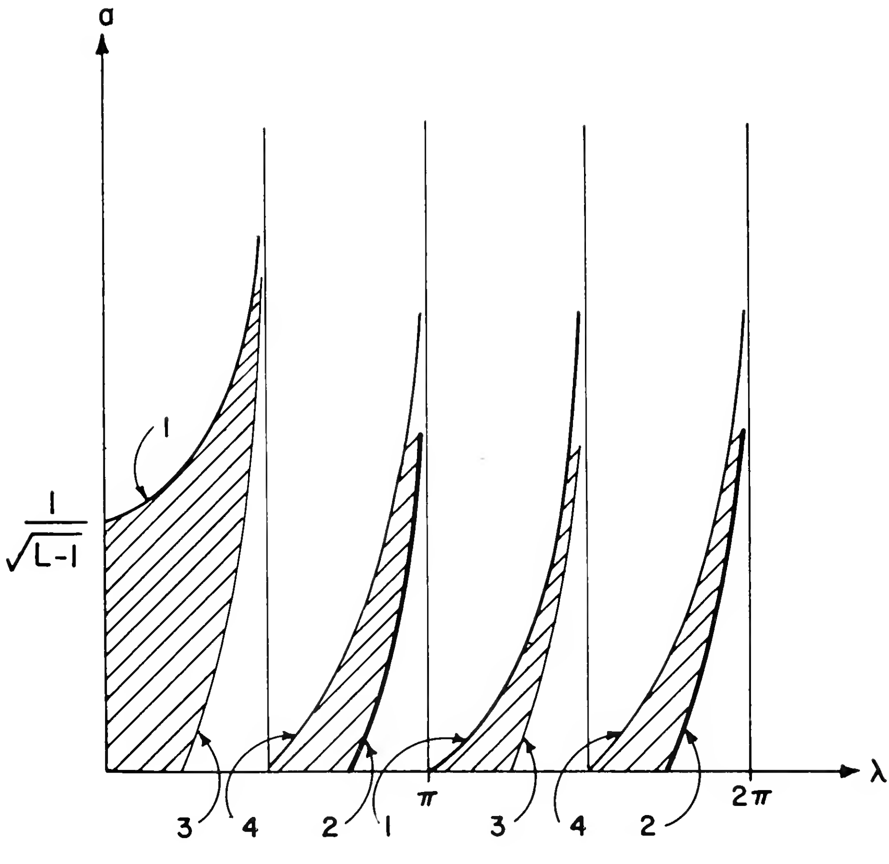

Figure 5

6. Applications

The equation discussed in Sections 2 and 3 has numerous physical applications. It can be derived from the problem of a vibrating string, composed of two homogeneous pieces. Another application is to the propagation of a wave in a one-dimensional structure, where the index of refraction is an even periodic, piecewise constant function.

Another application is to resonant inductance capacitance circuits, where the inductance or capacitance varies periodically with time. Such equations were treated approximately by Carson [4] and Minorsky [5].

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